This depressed equation may be solved for the other two roots of Eq. (1) by means of the quadratic formula.

References

[1] Cardan, G., *Ars Magna,* Nürnberg, 1545.
[2] Lyon, W.V., "Note on a Method of Evaluating the Complex Roots of Quartic Equations," *Journal of Math and Physics,* Vol. 3, April 1924, pp. 188-190.
[3] Jahnke, E. and Emde, F., *Tables of Functions,* 4th Ed., Dover, New York, 1945, Appendix, pp. 20-30.
[4] Graham, D. and Lathrop, R.C., "The Synthesis of 'Optimum' Transient Response: Criteria and Standard Forms," *Transactions of the AIEE,* Pt. 2, Vol. 72, Nov. 1953, pp. 273-288.
[5] Ashkenas, I.L. and McRuer, D.T., "Approximate Airframe Transfer Functions and Application to Single Sensor Control Systems," Wright-Patterson AFB, Ohio, WADC TR 58-82, June 1958, pp. 191-210.
[6] Neumark, S., *Solution of Cubic and Quartic Equations,* Pergamon Press, Oxford, England, 1965, pp. 5-11, 39-57.
[7] Dickson, L.E., *New First Course in the Theory of Equations,* Wiley, New York, 1939, pp. 10-12, 42-44.
[8] McRuer, D.T., Ashkenas, I.L., and Graham, D., *Aircraft Dynamics and Automatic Control,* Princeton University Press, Princeton, N.J., 1973, pp. 112-148.
[9] Ince, E.L., *Ordinary Differential Equations,* Dover, New York, 1944, pp. 62-63.
[10] Courant, R., *Differential and Integral Calculus,* New Revised Ed., Vol. I, Nordeman Publishing Co., New York, 1937, pp. 355-357, 359.

AIAA 81-4005

Geometric Dilution of Precision in Global Positioning System Navigation

Bertrand T. Fang*
Computer Sciences Corporation, Silver Spring, Md.

Introduction

THE NAVSTAR global positioning system (GPS), when fully operational in the early 1990's, will provide worldwide navigation through synchronized transmissions from a constellation of eighteen 12-h period satellites in three 55-deg-inclination orbital planes. An accurate user navigation fix (position and time) will be obtainable by receiving transmissions from four satellites and decoding the signal transit times.

One may relate the measurements, referred to as the pseudoranges, to the navigation state as follows:

$$CT_j=\sqrt{(X_1-x_1)^2+(X_2-x_2)^2+(X_3-x_3)^2}+X_4+n_j \tag{1}$$

where C is the velocity of light; T_j is the signal transit time from GPS satellite j to user (not corrected for user clock offset Δt); X_1,X_2,X_3,X_4 are user navigation states, where the first three represent a set of convenient cartesian user coordinates, and $X_4=C\Delta t$ is a range bias equivalent of user clock offset; x_1,x_2,x_3 are the corresponding cartesian coordinates of GPS satellite j; and n_j is random measurement noise.

From a set of four measurements, a user navigation fix may be determined. The accuracy of the fix is characterized by the following 4×4 position-time navigation error covariance matrix:

$$P=(H^TWH)^{-1} \tag{2}$$

where

$$H^T=\left[\begin{array}{c|c|c|c} a & b & c & d \\ \hline 1 & 1 & 1 & 1 \end{array}\right] \tag{3}$$

is the measurement partial derivative matrix, transposed; a, b, c, and d are line-of-sight unit vectors from a set of four GPS satellites to the user; W is the 4×4 covariance matrix of random measurement noise; and superscript T indicates the transpose of matrix.

The measurement error covariance matrix W is generally taken to be diagonal, which is strictly true for uncorrelated measurements only. In practice, assignments of quantitative values to the elements of W also take into consideration such factors as the elevation and health status of individual GPS satellites. Thus, W may more appropriately be referred to as the weighting matrix. For uniform weighting, P is proportional to $(H^TH)^{-1}$, which depends only on the relative geometry of the user and the four GPS satellites, as is evident from Eq. (3). The square root of the trace of $(H^TH)^{-1}$ is referred to as Geometric Dilution of Precision (GDOP), a self-explanatory name. Whatever the weighting strategy, the trace of the navigation error covariance matrix serves as a convenient and natural performance index characterizing the accuracy of the navigation fix. For a diagonal weighting matrix W, the trace of P is the sum of diagonal terms of $(H^TH)^{-1}$ weighted by the inverses of the corresponding elements of W.

Thus, the evaluation of the GPS navigation performance is essentially equivalent to the computation of the diagonal terms of $(H^TH)^{-1}$, which may be called the GDOP matrix for covenience.

The navigation performance index, $\mathrm{tr}(P)$, also serves as a criterion for the selection of a set of four best GPS satellites among those visible, which may be as many as ten for users which are satellites themselves. If, for optimum performance, each of the different combinations of four has to be evaluated, the computational burden can be considerable. In the following, certain theoretical results concerning the general properties of the GDOP matrix are derived. An efficient algorithm for the computation of the GDOP matrix and the navigation performance index is given; applications of the results are illustrated by numerical examples.

Analytical Results

To solve for a navigation fix from four measurements, the partial derivative matrix H must be nonsingular. Since the determination of H

$$|H|=\left|\begin{array}{c|c|c|c} a-d & b-d & c-d & d \\ \hline 0 & 0 & 0 & 1 \end{array}\right| = |a-d \mid b-d \mid c-d|$$

a navigation fix can be determined from four GPS satellites with line-of-sight directions a, b, c, and d if and only if the three vectors $(a-d)$, $(b-d)$, and $(c-d)$ are linearly independent, i.e., noncoplanar.

From Eq. (3) and the fact that a, b, c, and d are unit vectors, one obtains

$$\mathrm{tr}(H^TH)^{-1}=\mathrm{tr}(HH^T)^{-1}$$

$$=\mathrm{tr}\begin{bmatrix} 2 & a^Tb+1 & a^Tc+1 & a^Td+1 \\ & 2 & b^Tc+1 & b^Td+1 \\ & & 2 & c^Td+1 \\ \text{Symmetric} & & & 2 \end{bmatrix}^{-1} \tag{4}$$

Received May 27, 1980; revision received Aug. 25, 1980.

*Senior Principal Engineer, Orbit Operations, System Sciences Division.

A great deal of information about $\mathrm{tr}(HH^T)^{-1}$ may be obtained by examining the eigenvalues λ_i of the matrix HH^T appearing in Eq. (4). It is obvious that $\lambda_1+\lambda_2+\lambda_3+\lambda_4=8$. Let the λ's be ordered such that $0<\lambda_1\leq\lambda_2\leq\lambda_3\leq\lambda_4<8$. From the theorem of root separation for symmetric matrices,[1] one obtains $\lambda_1\leq 1-\cos\beta\leq\lambda_3$ and $\lambda_2\leq 3+\cos\beta\leq\lambda_4$, where $1-\cos\beta$ and $3+\cos\beta$ are eigenvalues of the 2×2 principal submatrices of HH^T with $\cos\beta$ equal to a^Tb or the scalar product of any two line-of-sight vectors. From the preceding properties of λ's and $\mathrm{tr}(HH^T)^{-1}=1/\lambda_1+1/\lambda_2+1/\lambda_3+1/\lambda_4$, the following bounds on the navigation performance index may be obtained:

$$\mathrm{tr}(H^TH)^{-1}>2 \tag{5}$$

$$3/\lambda_1+\tfrac{1}{2}>\mathrm{tr}(H^TH)^{-1}>1/\lambda_1+9/8 \tag{6}$$

$$\mathrm{tr}(H^TH)^{-1}>9/8+1/(1-\cos\theta) \tag{7}$$

where θ is the smallest angle subtended by any two line-of-sight vectors. Inequality (5) tells us the navigation performance index must exceed the value of 2. It will be shown later that for an artificial but not improbable GPS configuration, a performance index as low as 2.5 may be achieved. Inequality (6) says that knowledge of the smallest eigenvalue of HH^T gives us both lower and upper bounds on the performance index. Frequently, that lower bound also serves as a good estimate. The lower bound provided by Eq. (7), although not as sharp as that of Eq. (6), is easy to calculate and the most useful. It expresses the intuitive rule of thumb that accurate navigation should not rely on a GPS constellation that is clustered together. In particular, it follows from Eq. (7) that a navigation performance index in excess of 8.5 would result if any two line-of-sight vectors to GPS satellites are separated by 30 deg or less.

Algorithm

Since $\mathrm{tr}(H^TH)^{-1}=\mathrm{tr}(H^{-1}(H^{-1})^T)=$ sum of the squares of the elements of H^{-1}, the computation of the navigation performance index is essentially the inversion of the 4×4 measurement partial derivative matrix H. In testing different combinations of visible GPS satellites to obtain a best set of four, many inversions may have to be conducted. On the other hand, the different combinations often have some satellites in common, which means the different H's have rows in common, and this fact may be taken advantage of in performing matrix inversions. The following algorithm assumes the switch of GPS satellites one at a time.

Consider that for a particular GPS configuration, $H^{-1}=G$ is already obtained. If the nth $(n=1,2,3,4)$ GPS satellite with line-of-sight vector r is to be replaced by another satellite with line-of-sight vector p, the new measurement partial derivative matrix may be written as

$$H'=H+\begin{bmatrix}\delta_{1n}\\ \delta_{2n}\\ \delta_{3n}\\ \delta_{4n}\end{bmatrix}[(p-r)^T \,\vdots\, 0] \tag{8}$$

where δ_{in} is the Kronecker delta. From the preceding decomposition, the Sherman-Morrison formula[2] gives us

$$H'^{-1}=G-\begin{bmatrix}G_{1n}\\ G_{2n}\\ G_{3n}\\ G_{4n}\end{bmatrix}\left[(p-r)^T\begin{pmatrix}G_{11} & G_{12} & G_{13} & G_{14}\\ G_{21} & G_{22} & G_{23} & G_{24}\\ G_{31} & G_{32} & G_{33} & G_{34}\end{pmatrix}\right]\times\left\{1+[p-r]^T\begin{bmatrix}G_{1n}\\ G_{2n}\\ G_{3n}\end{bmatrix}\right\}^{-1} \tag{9}$$

The computational economy provided by this equation is obvious.

One may also use the previous algorithm to reduce the initial inversion of the 4×4 measurement partial derivative matrix to an inversion of its 3×3 submatrix. One way to accomplish this is to set $r=0$ and $n=4$ in Eq. (8). A little bit of algebra would result in the following expression for the navigation performance index:

$$\begin{aligned}\mathrm{tr}(HH^T)^{-1}&=f^Tf+g^Tg+h^Th\\ &+(2/\alpha)\{(d^Tf)q^Tf+(d^Tg)q^Tg+(d^Th)q^Th\}\\ &+(q^Tq+1/\alpha^2)\{1+(d^Tf)^2+(d^Tg)^2+(d^Th)^2\}\end{aligned} \tag{10}$$

where $(f\,\vdots\,g\,\vdots\,h)^T=(a\,\vdots\,b\,\vdots\,c)^{-1}$, $q=f+g+h$ and $\alpha=1-d^Tq$; and a, b, and c are line-of-sight vectors to the first three GPS satellites. When the fourth satellite is switched, it is only necessary to change the line-of-sight vector d in Eq. (10) in order to obtain the performance index of the new configuration.

Applications

Intuitively, orthogonal line-of-sight user-to-GPS satellite configurations are favorable. In three-dimensional space, it is, of course, impossible to have a set of four mutually orthogonal unit vectors. An alternative has three of the line-of-sight vectors a, b, and c orthogonal. For this case, the vectors f, g, and h become the same orthogonal unit vectors as a, b, and c, and Eq. (10) simplifies to

$$\begin{aligned}\mathrm{tr}(HH^T)^{-1}&=3+2(d_1+d_2+d_3)/\alpha+4(1+d_1^2+d_2^2+d_3^2)/\alpha^2\\ &=1+\{10-2(d_1+d_2+d_3)\}/\{1-(d_1+d_2+d_3)\}\end{aligned}$$

where d_1, d_2, and d_3 with $d_1^2+d_2^2+d_3^2=1$ are components of the line-of-sight unit vector d along the orthogonal a, b, and c directions. It is of interest to note that for this case, the navigation performance index depends only on $(d_1+d_2+d_3)$, the simplest symmetric function of the components of the vector d. The best performance index of 2.80 is achieved for $d^T=(-1,-1,-1)/\sqrt{3}$. This is the situation that the line-of-sight to the GPS satellite "d" shows no preference to, but is directed away from the other GPS satellites, an artificial but not improbable configuration for a user satellite. For $d^T=(1,1,1)/\sqrt{3}$, i.e., d having the same general direction as the other three lines-of-sight, the performance index degrades to 13.20. This degradation reminds us of the statement made earlier about avoiding closely grouped GPS satellites. For $d=-a$, i.e., for a user located between two GPS satellites, the performance index has the value 4.00.

There is reason to believe that a GPS constellation with $a-d$, $b-d$, and $c-d$ orthogonal may give good navigation performance. This may be realized with the set of line-of-sight vectors $a^T=(-1,1,1)/\sqrt{3}$, $b^T=(1,-1,1)/\sqrt{3}$, $c^T=(1,1,-1)/\sqrt{3}$, and $d^T=(1,1,1)/\sqrt{3}$. However, for this configuration, the angle between the vector d and any other vector is $\cos^{-1}(2/3)$, which is comparatively small and may be undesirable from the consideration of the preceding section. Indeed, it follows immediately from inequality (7) that the navigation performance index must be in excess of $9/8+1/(1-2/3)=4\tfrac{1}{8}$, a lower bound which may be compared with the exact index of 5.5, obtainable from straightforward simple computation. On the other hand, by reversing the direction of the vector d previously given, one has the completely symmetrical configuration that the line-of-sight vectors are all separated by the same angle $\cos^{-1}(-1/3)$. For this configuration, one may compute the eigenvalues of HH^T as $\lambda_1=\lambda_2=\lambda_3=4/3$ and $\lambda_4=4$, giving rise to a navigation performance index $\mathrm{tr}(HH^T)^{-1}=2.5$ Notice that $\lambda_1=4/3$ achieves the upper bound $1-\cos\beta$ discussed earlier. Since any perturbation of the configuration will result in a decrease in the minimum angle between two line-of-sight vectors, and therefore a decrease in λ_1, this configuration maximizes the smallest eigenvalue of HH^T. Whether this also happens to be the best configuration remains to be investigated.

Acknowledgments

This work was sponsored by NASA Goddard Space Flight Center under Contract No. NAS 5-24300. The author is in debt to Dr. Dale Headrick of Computer Sciences Corporation for pointing out the completely symmetrical configuration described herein.

References

[1] Pearson, C.E., (ed.), *Handbook of Applied Mathematics*, 1st Ed., Van Nostrand, New York, 1974, p. 926.

[2] Householder, A.S., *Principles of Numerical Analysis*, McGraw-Hill, New York, 1953, pp. 78-79.

Technical Comments

AIAA 81-4006

Comment on "Formulation of Equations of Motion for Complex Spacecraft"

Carl Grubin*

Teledyne Systems Company, Northridge, Calif.

Kane and Levinson[1] have written a very comprehensive paper which compares various techniques for formulating the equations of motion for complex systems of moving masses. In general their analysis is quite commendable, however, there are two points worthy of comment: 1) In formulating the angular momentum equations it is not necessary to locate explicitly the combined mass center (CMC) for the entire system. Thus by doing a few preliminary vector calculations, Eq. (10KL)† can be derived directly rather than arriving at Eq. (9KL) and then algebraically simplifying this to Eq. (10KL). 2) The origin for the angular momentum equation is not restricted to the CMC; any origin can be used provided an additional term is added. This "generalized angular momentum equation" effectively constitutes the proof of D'Alembert's principle. These points will now be elaborated upon.

Received May 27, 1980; revision received Sept. 2, 1980.

*Research Scientist. Associate Fellow AIAA.

†Equation or Fig. numbers followed by KL refer to those in Ref. 1.

‡Since there is only a single mass m moving relative to the main body, the "1" subscript can be dropped.

1) The angular momentum for a system of bodies with respect to their CMC can be written as the sum of the angular momenta for each body with respect to its own mass center (MC) plus transfer terms of the form

$$m_r\bar{\rho}_r\times\dot{\bar{\rho}}_r \qquad (r=0,1,2,\ldots,n) \tag{1}$$

where m_r is the mass of body r (there are $n+1$ bodies), $\bar{\rho}_r$ is the position vector of the MC of body r relative to the CMC, and $\dot{\bar{\rho}}_r=\mathrm{d}\bar{\rho}_r/\mathrm{d}t$, where $\mathrm{d}/\mathrm{d}t$ is the time derivative taken with respect to a nonrotating coordinate system centered at the CMC. In these "moving parts" type problems $\bar{\rho}_r$ is usually specified with respect to the MC of the main body B_0 by a relative position vector $\bar{q}_r$. Thus if $\bar{\rho}_0$ is the position vector from the CMC to the MC of B_0

$$\bar{\rho}_r=\bar{\rho}_0+\bar{q}_r \tag{2}$$

By definition of the CMC

$$\sum_{r=0}^{n} m_r\bar{\rho}_r=0 \tag{3}$$

Eliminating $\bar{\rho}_0$ between Eqs. (2) and (3) determines $\bar{\rho}_r$ (and $\dot{\bar{\rho}}_r$) as

$$\bar{\rho}_r=\bar{q}_r-\frac{1}{\Sigma m}\sum_{r=1}^{n} m_r\bar{q}_r \qquad (r=0,1,2,\ldots,n;\ \bar{q}_0\equiv 0) \tag{4}$$

where

$$\Sigma m=\sum_{r=0}^{n} m_r$$